SPECIALLY WEIGHTED MOVING AVERAGES WITH REPEATED APPLICATION OF THE EMA OPERATOR

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UAM.1991-10-14

December 7, 2000

An internal paper by the O&A Research Group

Contents

1 INTRODUCTION

The *moving average* (MA) is a useful tool to summarize the past behavior of a time series at any time point. MAs are used for different purposes, such as forecasts and trading models. In many cases, they are used in form of *momenta*, that is differences of the current time series value and an MA or between two different MAs. Moving averages can be defined with different *weighting functions* of their summation. The choice of the weighting function has an influence on the *success* of the MA in its application. Studying moving averages with different weighting functions is thus a relevant research task.

Among the MAs, the *exponentially weighted moving average* (EMA) plays an important role. Its weighting function declines exponentially with the time distance of the past observations from now. The sequential computation of EMAs along a time series is simple with the help of a *recursion* formula; it is more efficient than the computation of any differently weighted MA. The EMA computation has been documented in (Müller et al., 1987); the present paper is based on the newer document (Müller, 1989). EMA applications have been found useful in many empirical studies. For time series with a strong random element, however, the rapidly increasing shape of the exponential function leads to a strong weight of the very recent past and hence for short-term noise structures of the time series. This seems to be a reason for the fact that other MA weighting functions have been found more successful in applications.

In some recent trading model tests of (Dacorogna, 1991), two new families of MA weighting functions have been found remarkably successful. Both families can be developed with *repeated application* of the EMA *operator*. The EMA can be regarded as an operator on the time series that yields a new time series: the series of the EMAs. Since the result of this operator has the same mathematical structure as its argument, that of a time series, it can be applied arbitrarily many times by taking the result of the previous application as the argument of the new one.

This paper presents the computation and some properties of the new MA families. In section 2, some basic definitions of MAs and EMAs are given. The idea of EMA operators, introduced in section 3.1, is used for defining and discussing the new MA family $EMA^{(i)}$ in sections 3.2 and 3.3 and the family $EMA^{(j,n)}$ in section 3.4.

2 BASIC DEFINITIONS

2.1 Moving averages

Basic definitions and formulas for moving averages are given in (Müller, 1989). Some of these are repeated here in order to give complete information within one paper.

A moving average of the time series x_i (at time points t_i whose sequence is monotonously increasing but may be *unequally spaced*) is a weighted average of the series elements of the past up to "now", t_n. The standard meaning of x_i is the series of *logarithmic middle prices*, see (Müller et al., 1987) or (Müller et al., 1990), but the formulas presented here can be used for any time series. The time points t_i may be defined on any time scale, not only physical time.

The time series x_i can be expanded to a time *function* $x(t)$ by *interpolating* between the series elements. Different interpolation methods can be applied; four of them are considered in this paper:

1. linear interpolation between series elements;
2. always taking the x_j value of the *preceding* series element.
3. always taking the x_j value of the *nearest* series element (measured on the t axis);
4. always taking the x_j value of the *subsequent* series element.

The linear interpolation is probably the best choice; the three last interpolation methods, composed of constant pieces and sudden jumps, assume an unrealistically *noisy* behavior of x_j. The second interpolation that takes the preceding series element for the whole interval is appropriate for *other* applications, but usually not for computing moving averages, as it leads to values at t_n which are *not* influenced by the *most recent* series value x_n.

Such an interpolation is almost inevitable to make *unequally spaced* time series tractable. For the function $x(t)$, the moving average at the time t_n is defined as an integral,

$$MA_{x,w}(t_n) \quad \equiv \quad \frac{\int_{-\infty}^{t_n} w(t_n - t)\ x(t)\ dt}{\int_{-\infty}^{t_n} w(t_n - t)\ dt} \quad . \tag{2.1}$$

The *weighting function* $w(\Delta t)$ is defined only for positive and zero arguments. Its integral in the denominator, which does not depend on t_n, must of course exist ($< \infty$).

Equally spaced series can also be expanded to a function $x(t)$. In this case, however, the alternative is to keep a genuine *discrete* time series, particularly if the definition of values *between* the series elements is "unnatural" for the time series. The definition for the moving average of a regular discrete time series is

$$MA_{x,w;n} \quad \equiv \quad \frac{\sum_{j=-\infty}^{n} w_{n-j}\ x_j}{\sum_{j=-\infty}^{n} w_{n-j}} \quad . \tag{2.2}$$

w_k is a series of weights independent of n. The sum in the denominator again has to be finite.

A fundamental property of a moving average is its *range* Δt_r. It is defined as the center of gravity of the weighting function $w(\Delta t)$,

$$\Delta t_r \quad \equiv \quad \frac{\int_{-\infty}^{t_n} w(t_n - t)\ (t_n - t)\ dt}{\int_{-\infty}^{t_n} w(t_n - t)\ dt} \quad \equiv \quad \frac{\int_0^{\infty} w(\Delta t)\ \Delta t\ d(\Delta t)}{\int_0^{\infty} w(\Delta t)\ d(\Delta t)} \quad ; \tag{2.3}$$

or, for an equally spaced discrete series,

$$r \quad \equiv \quad \frac{\sum_{j=-\infty}^{n} w_{n-j}\ (n-j)}{\sum_{j=-\infty}^{n} w_{n-j}} \quad \equiv \quad \frac{\sum_{k=0}^{\infty} w_k\ k}{\sum_{k=0}^{\infty} w_k} \quad . \tag{2.4}$$

The range r of a discrete series is in units of the time series index i, but unlike this integer index it can be any positive real number.

2.2 Exponentially weighted moving averages

Exponential moving averages (EMAs) have the following exponentially declining weights:

$$w(\Delta t) \equiv e^{-\frac{\Delta t}{\Delta t_r}} \quad , \tag{2.5}$$

and, for an equally spaced discrete series,

$$w_k \equiv (\frac{r}{r+1})^k \quad . \tag{2.6}$$

This geometric sequence results from an initial formula, $w_k = \exp(-ck)$, inserted in equation 2.4, by using the sum formula 0.231.2 of (Gradshteyn and Ryzhik, 1980).

These equations for the weights are formulated by using the *range* (see equations 2.3 and 2.4) as the main parameter. This was done on purpose, in order to provide a *type-independent* parameter for moving averages. If we *substitute* a type of moving average by another one, we should always choose the one with the *same range*! This principle is not carefully observed in some literature references.

An EMA can be computed by a *recursion* formula. If its value $EMA_x(\Delta t_r, t_{n-1})$ at the time t_{n-1} of the previous series element x_{n-1} is known, we can easily compute the value at t_n. For all interpolation methods and also for an equally spaced discrete series, the following recursion formula can be used:

$$EMA_x(\Delta t_r, t_n) = \mu \; EMA_x(\Delta t_r, t_{n-1}) + (1-\mu)\; x_n + (\mu - \nu)\; \Delta x_n \quad , \tag{2.7}$$

with

$$\Delta x_n \equiv x_n - x_{n-1} \quad . \tag{2.8}$$

For a time series representing a function with *any* type of interpolation, we get

$$\mu = e^{-\alpha} = e^{-\frac{\Delta t_n}{\Delta t_r}} \quad , \tag{2.9}$$

similar to equation 2.5, with

$$\alpha \equiv \frac{\Delta t_n}{\Delta t_r} \quad , \tag{2.10}$$

and the time interval

$$\Delta t_n \equiv t_n - t_{n-1} \quad . \tag{2.11}$$

For an equally spaced discrete series, we obtain a *different* factor μ:

$$\mu = \frac{r}{r+1} \quad , \tag{2.12}$$

similar to equation 2.6.

The variable ν depends on the series type and the interpolation method:

$$\nu = \begin{cases} \frac{1-e^{-\alpha}}{\alpha} = \frac{1-\mu}{\alpha} & \text{for the linear interpolation} \\ 1 & \text{for taking the preceding series value} \\ e^{-\frac{\alpha}{2}} = \sqrt{\mu} & \text{for taking the nearest series value} \\ e^{-\alpha} = \mu & \text{for taking the subsequent series value} \\ \frac{r}{r+1} = \mu & \text{for equally spaced discrete time series (no interpolation).} \end{cases} \tag{2.13}$$

The relation $\nu = \mu$, valid for the discrete series case, is surprisingly also true for the *least* appealing interpolation method, that taking the *subsequent* series value. This relation makes the last term of the recursion 2.7 vanish.

Recursions are not enough for computing exponential moving averages. They need an initial value to start with. There is usually no information before the first series element x_1 which is the natural choice for this initialization:

$$EMA_x(\Delta t_r, t_1) = x_1 \quad . \tag{2.14}$$

The error made by this initialization declines with the factor $e^{-(t_n - t_1)/\Delta t_r}$, respectively $[r/(r+1)]^{n-1}$ for a discrete series. In many applications, the EMA will not yet be used at t_1 and in the initial phase after t_1 which is called the *build-up* time. After the build-up time, when we start using the EMA, we expect it to be (almost) free of initialization errors.

How long should an appropriate build-up time be? If high precision is required, the initial error should be reduced to the granularity of the real number representation of the computer. For the 64 bit real numbers used in the O&A environment, this is about 10^{-16} times the EMA value. If we assume an initial error of the same order of magnitude as the EMA, then we need a long build-up time of about $35\Delta t_r$. In most cases, we can take a smaller build-up time, because the initial error is smaller than the EMA and the relative error of the x observations is much greater than 10^{-16}.

2.3 Momenta

Moving averages are often used indirectly, through *momenta*. The general definition of a momentum is

$$m_{x,w}(t_n) \equiv x_n - MA_{x,w}(t_n) \quad , \tag{2.15}$$

where $MA_{x,w}$ can be any type of a moving average of the series x_i.

Again, we are most interested in exponential moving averages and thus "exponential momenta". The general recursion formula for them is

$$m_x(\Delta t_r, t_n) \quad = \quad \mu \; m_x(\Delta t_r, t_{n-1}) + \nu \; \Delta x_n \quad . \tag{2.16}$$

This recursion is slightly *simpler* than the one for exponential moving averages (equation 2.7). Δx_n, μ, and ν are still defined by the equations 2.8, 2.9, 2.12, and 2.13.

Recursions for momenta need an initialization at the time t_1, consistent to equation 2.14,

$$m_x(\Delta t_r, t_1) \quad = \quad 0 \quad . \tag{2.17}$$

The error made by this initialization again declines with the factor $e^{-(t_n - t_1)/\Delta t_r}$, respectively $[r/(r+1)]^{n-1}$ for a discrete series. Thus, momenta need a similarly long build-up time as EMAs, about $35 \Delta t_r$ if high precision is required.

The momentum of the equations 2.15 and 2.16 is the same as the "simple momentum" of (Müller et al., 1987). This document also presents other types of momenta, the *first* and the *second* momentum. The first momentum is the difference of two exponential averages (or momenta) with *different ranges*. The second momentum is the linear combination of three exponential moving averages (or momenta) with different ranges; it indicates the overall *curvature* of the series curve for a certain depth of the past. Both are described in (Müller et al., 1987) and can be computed by existing O&A software. They vary smoother with time than the more noisy simple momenta.

It is hard to judge the value of a momentum unless there is something to compare with. For logarithmic FX prices, the *scaling law* (see (Müller et al., 1990)) can be used. Momenta can be regarded as exponential moving averages of price *changes*, so the scaling law is *approximately* valid for momenta, too. A useful variable is the *relative* momentum,

$$M_x(\Delta t_r, t_n) \quad = \quad \frac{m_x(\Delta t_r, t_n)}{c \; \Delta t_r^D} \quad , \tag{2.18}$$

scaled by the scaling law for its *range*. The denominator can be obtained through O&A software: the Modula-2 procedure `TimeScale.ComputeScaling` or the G function `scale-price`. It is better to use the time scale ϑ instead of physical time t for this.

3 REPEATED APPLICATION OF THE EMA OPERATOR

3.1 The EMA as an operator

The exponential moving average $EMA_x(\Delta t_r, t_n)$ is a time series as well as the original series $x(t_n)$. The EMA can thus be regarded as an *operator* EMA that transforms one time series into another:

$$\mathrm{EMA}: \; x(t_n) \quad \longmapsto \quad \mathrm{EMA}[x(t_n)] \quad \equiv \quad EMA_x(\Delta t_r, t_n) \quad . \tag{3.1}$$

For the further discussion, it is useful to write the EMA definition, eq. 2.1, for a time series expanded to the function $x(t)$ together with the weights of eq. 2.5,

$$\text{EMA}: \; x(t) \;\longmapsto\; \text{EMA}[x(t)] \;\equiv\; EMA_x(\Delta t_r, t) \;=\; \frac{1}{\Delta t_r} \int_{-\infty}^{t} e^{-\frac{t-t'}{\Delta t_r}} \; x(t') \; dt' \quad . \tag{3.2}$$

3.2 Repeated application of the same EMA operator

The EMA operator yields a result of the same mathematical nature as its input: a time series or, when expanded as in eq. 3.2, a time function. Thanks to this property, the EMA operator can be applied *another* time to the result of its first application. The result will again be a moving average (MA) in the sense of eq. 2.1, but with a weighting function more complicated than a simple exponential function. Obtaining new weighting functions is indeed a motivation to study the repeated application of EMA operators. One of the main advantages of EMAs, their efficient recursion formula (eq. 2.7), is preserved also for repeated applications.

In principle, the *ranges* Δt_r of the sequentially applied EMA operators can differ. The variety of new MAs based on different EMA ranges is larger than if all EMA operators have the same range. In this paper, however, the emphasis lies on EMA operators with the *same* range Δt_r, for two reasons. First, they are more efficient in their computation as the constants μ and ν can be shared between the EMA operators (see eqs. 2.9, 2.12, and 2.13); second, the MA family generated with them has particularly interesting and useful properties.

The weighting function obtained by applying the EMA operator *twice* can be derived from eq. 3.2,

$$\text{EMA}: \; EMA_x(\Delta t_r, t) \;\longmapsto\; \text{EMA}[EMA_x(\Delta t_r, t)] \;\equiv\; EMA_x^{(2)}(\Delta t_r, t) \;=$$

$$\frac{1}{{\Delta t_r}^2} \int_{-\infty}^{t} (t - t') \; e^{-\frac{t-t'}{\Delta t_r}} \; x(t') \; dt' \quad , \tag{3.3}$$

where the integral has been computed with product rule of integration. The notation $EMA^{(2)}$ refers to the number of applications of the EMA operator. Hence, $EMA^{(1)}$ is the normal EMA and $EMA_x^{(0)}$ is the unchanged original function $x(t)$.

The above step can be repeated to obtain the general case $EMA^{(n)}$ for any positive integer n:

$$EMA_x^{(n)}(\Delta t_r, t) \;=\; \frac{1}{(n-1)! \; {\Delta t_r}^n} \int_{-\infty}^{t} (t - t')^{n-1} \; e^{-\frac{t-t'}{\Delta t_r}} \; x(t') \; dt' \quad . \tag{3.4}$$

This definition can also be written with the help of the weighting function $w(\Delta t)$,

$$EMA_x^{(n)}(\Delta t_r, t) \;=\; \int_{-\infty}^{t} w(t - t') \; x(t') \; dt' \quad , \tag{3.5}$$

where

$$w(\Delta t) \;=\; \frac{\Delta t^{\; n-1}}{(n-1)! \; {\Delta t_r}^n} \; e^{-\frac{\Delta t}{\Delta t_r}} \quad . \tag{3.6}$$

The integral of this function $w(\Delta t)$ from 0 to ∞ is already normalized to the value 1.

Before discussing the properties of the new MAs, a problem of their computation has to be addressed. The above formulas are valid for *really* continuous functions $x(t)$. In our case of *interpolated* time series, however, the *first* application of the EMA operator yields a function *more complicated* than linear *between* the time series elements, no matter what interpolation method we assumed for $x(t)$. When applying the EMA operator a *second* time, we are, strictly speaking, obliged to use this complicated $EMA_x(\Delta t_r, t)$ function instead of a simple interpolation function supported in eq. 2.13. Using the recursion, eq. 2.7, with eq. 2.13 in repeated applications introduces a certain error. This interpolation problem mainly matters in case of data holes and low data density. If the data interval Δt_n is small compared to Δt, the error is very small. Heuristic investigations have shown that this error is harmless especially if *linear* interpolation is assumed in the recursions of all EMA operators.

In the case of discrete, equally spaced time series, the EMA recursion of eq. 2.7 can also be applied, without any problems of interpolation between the points.

Thus, the recursion of eq. 2.7 can be used n times to compute $EMA^{(n)}$. As intermediary results, we obtain the MAs $EMA^{(i)}$ for all $i < n$. All these intermediary MAs have to be initialized at the beginning of the time series, t_1:

$$EMA_x^{(i)}(\Delta t_r, t_1) \quad = \quad x_1 \quad , \tag{3.7}$$

analogous to eq. 2.14. The error of an individual $EMA^{(i)}$ declines with the factor $e^{-(t_n - t_1)/\Delta t_r}$, respectively $[r/(r+1)]^{n-1}$ for a discrete series. The error of $EMA^{(i)}$, however, has a chance of really declining only when $EMA^{(i-1)}$ has already reached its almost correct level. As a rule of thumb, we conclude that the error of $EMA^{(n)}$ needs a time of n times that of $EMA^{(1)}$ to decline. A discussion of the build-up time in the next section, however, shows that a higher $EMA^{(n)}$ does not need such a long build-up as a simple EMA with the *same center of gravity* (see also the build-up discussion at the end of section 2.2).

3.3 The properties of $EMA^{(n)}$

A moving average is characterized by its *range*, the center of gravity of its weighting function. The range Δt_r used in the recursions is the range of only $EMA^{(1)}$; the other $EMA^{(n)}$ have other ranges. The real range Δt_R of $EMA^{(n)}$ can be computed with the help of eqs. 2.3 and 3.6,

$$\Delta t_R \quad = \quad n \, \Delta t_r \quad . \tag{3.8}$$

This simple law can be visualized as follows: each application of the EMA operator pushes the range back by another Δt_r interval.

Such a behavior is, by the way, also found for repeated applications of EMA operators with *different* ranges Δt_r: each application pushes the center of gravity back by the Δt_r value of the individual EMA operator. Another property of these MAs is the *commutativity* of different EMA operators. In the case of $EMA^{(n)}$, this commutativity does not matter as the EMA operators are anyway equal.

Another important property of a weighting function $w(\Delta t)$ is its *width*. A user of a certain moving average wants to know if it concentrates all its weight on a narrow time interval or is influenced by many events over a long time span. The obvious statistical variable to measure the width is the *variance* Δt_V^2 about the center of gravity Δt_R. In the sense of eq. 2.3, we define

$$\Delta t_V^2 \quad \equiv \quad \frac{\int_0^\infty w(\Delta t)\ (\Delta t - \Delta t_R)^2\ d(\Delta t)}{\int_0^\infty w(\Delta t)\ d(\Delta t)} \quad . \tag{3.9}$$

The square root of the variance, the *standard deviation* Δt_V, is a natural parameter to describe the width of the weighting function.

By inserting eq. 3.8 and the weighting function of eq. 3.6, the variance for $EMA^{(n)}$ can be computed:

$$\Delta t_V^2 \quad = \quad n\ \Delta t_r^{\,2} \quad = \quad \frac{1}{n}\ \Delta t_R^2 \quad , \tag{3.10}$$

and the standard deviation,

$$\Delta t_V \quad = \quad \sqrt{n}\ \Delta t_r \quad = \quad \frac{1}{\sqrt{n}}\ \Delta t_R \quad . \tag{3.11}$$

On the extreme right hand side of this equation, the standard deviation is expressed in terms of the range Δt_R. This shows that, for large n, the standard deviation becomes *small* compared to the range, so the weighting function $w(\Delta t)$ attains the form of a narrow peak far from $\Delta t = 0$.

The forms of the weighting functions $w(\Delta t)$ of the different $EMA^{(n)}$ can be described as follows:

- For $EMA_x^{(0)} = x$, the weighting function is a δ distribution that concentrates the full weight at the current moment ($\Delta t = 0$).
- $EMA^{(1)} = EMA$ has an exponentially declining weighting function as defined in eq. 2.5 or eq. 2.6.
- $EMA^{(2)}$ has a $w(\Delta t)$ with a zero at $\Delta t = 0$ and a non-zero slope. $w(\Delta t)$ reaches a maximum and then asymptotically declines to zero.
- The weighting function of $EMA^{(3)}$ is like that of $EMA^{(2)}$, but with the maximum at a larger Δt and a zero slope at $\Delta t = 0$.
- $EMA^{(n)}$ with a large n has a weighting function similar to a Gaussian bell curve centered around $\Delta t = \Delta t_R$ (see eq. 3.8) with the variance of eq. 3.10.

With the help of these findings, the build-up time needed for $EMA^{(n)}$ can now be assessed. In the case of simple EMAs ($EMA^{(1)}$), we need a long build-up of about $35\Delta t_R$ to make the initial error due to eq. 3.7 vanish and attain high precision. This has been explained at the end of section 2.2. In the opposite case of very high n, the weighting function approaches a Gaussian bell curve. For this case, we assess the build-up time to the theoretical 10^{-16} fractile of the Gaussian bell curve: this is the center of the bell curve minus roughly 8 standard deviations. By applying this to an $EMA^{(100)}$, we obtain, with the help of eq. 3.8 and eq. 3.10, a build-up time of only $1.8\Delta t_R$. In many practical cases, n is neither 1 nor 100, but somewhere in between, so we expect a necessary build-up time somewhere between $1.8\Delta t_R$ and $35\Delta t_R$. In the study of (Dacorogna, 1991), a build-up of $12\Delta t_R$ suffices to eliminate the initial error of $EMA^{(4)}$ with full computer precision.

3.4 Linear combinations of $EMA^{(i)}$

The repeated application of the EMA operator implies the computation of not only $EMA^{(n)}$, but also all $EMA^{(i)}$ with $i < n$. These other MAs are available without any additional computation. A new family of MAs can thus be *efficiently* computed: *linear combinations* of the $EMA^{(i)}$.

Among the different possible linear combinations of the $EMA^{(i)}$, there is the special class of linear combinations with *equal coefficients* for all the $EMA^{(i)}$ used, where these i are in an uninterrupted sequence:

$$EMA_x^{(j,n)}(\Delta t_r, t) \quad = \quad \frac{1}{n+1-j} \sum_{i=j}^{n} EMA_x^{(i)}(\Delta t_r, t) \quad , \quad \text{with } 0 \;<\; j \;<\; n \;. \tag{3.12}$$

The scaling factor $1/(n+1-j)$ is chosen so to make this new MA fulfill the MA definition of eq. 2.1. The moving average $EMA^{(j,n)}$, defined by eq. 3.12, has a remarkable property: its weighting function $w(\Delta t)$ has a *plateau* before it asymptotically declines to zero. This plateau always represents the maximum of the weighting function; the larger the difference $n-j$, the wider the plateau. In the case $j=1$, it starts already at the current moment, at $\Delta t = 0$. This can be demonstrated by taking the weighting functions from eq. 3.6 and deriving their sum against Δt. If $j > 1$, then $w(\Delta t)$ starts with low values in the vicinity of $\Delta t = 0$ and reaches the plateau only after a certain Δt distance.

The term “plateau”, however, should not always be understood literally: if the difference $n-j$ or rather the ratio n/j is small, then a term like “wide peak” describes the geometry of the weighting function more appropriately.

The plateau property seems to be favorable in applications, as shown for trading models in (Dacorogna, 1991). One possible reason for this lies in the geometric nature of a plateau: there is *no sharp peak* in the weighting function that can lead to a noisy MA behavior when acting on a noisy short-term price move.

Due to the linearity of the $EMA^{(i)}$ combination of eq. 3.12, the *range* Δt_R of $EMA^{(j,n)}$ can easily be derived. It is simply the mean of all $EMA^{(i)}$ ranges (see eq. 3.8) used in its computation:

$$\Delta t_R \quad = \quad \frac{n+j}{2}\, \Delta t_r \quad . \tag{3.13}$$

The variance Δt_V^2 of the weighting function can also be computed from the the individual Δt_R and Δt_V^2 values of the $EMA^{(i)}$ used in the computation, with the help of the law of moments.

For *very large* n, the weighting function of $EMA^{(1,n)}$ approaches a *rectangular* form: a flat plateau with a sudden cutoff at a certain Δt ($\approx 2\Delta t_R$). Such a rectangular MA is frequently applied in conventional *Technical Analysis* of market prices. For moderately high n, however, the weighting function is still far from rectangular: its transition period, where the value varies between 10% and 90% of the maximum, is roughly $2\Delta t_R/\sqrt{n}$. For $n = 4$, for example, this transition period is as large as Δt_R.

The *momenta* of section 2.3 can be regarded as linear combinations of $EMA^{(i)}$, though of a nature different from eq. 3.4:

$$m_x \equiv x - EMA_x = EMA_x^{(0)} - EMA_x^{(1)} \quad . \tag{3.14}$$

This leads to a not fully explored idea: applying momenta defined as differences between more complicated MAs.

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